- Homogeneous functions
- Bernoulli's equation
- $y' = f(Ax + By + C)$

(f) $\boxed{y' = f(x,y)}$, $f(x,y)$ homogeneous

Homogeneous functions:

$$f(tx, ty) = t^{\alpha} f(x,y), \quad \alpha \text{ is a constant}$$

$$\Longleftrightarrow f(x,y) = F\left(\frac{y}{x}\right)$$

Ex: $f(x,y) = \frac{y^2}{x^2} + \frac{y}{x}$ $\left\{u = \frac{y}{x}\right\} = u^2 + u = F(u)$

$$f(tx, ty) = \frac{\cancel{(ty)^2}}{(tx)^2} + \frac{ty}{tx} = \frac{y^2}{x^2} + \frac{y}{x} =$$

$$= t^0 f(x,y)$$

Homogeneous $\alpha = 0$.

Method of solving:

Sub: $\boxed{u = y/x}$

$y' = f(x,y) \Leftarrow$ homogenous

$$\boxed{u'x + u = F(u)}$$

$$u'x = F(u) - u.$$

$$\boxed{\frac{du}{F(u) - u} = \frac{dx}{x}} \Leftarrow \text{separable}$$

$u = \frac{y}{x}$, $y = ux$

$y' = u'x + u$

Ex : $y' = \underbrace{\frac{y^2}{x^2} + \frac{y}{x}}_{\text{Homogeneous}}$

$u = \frac{y}{x}$, $\quad u'x + \cancel{u} = u^2 + \cancel{u}$

$u'x = u^2 \qquad (F(u) - u)$

$\int \frac{du}{u^2} = \int \frac{dx}{x}$

$-\frac{x}{y} = -\frac{1}{u} = \ln|x| + c$

$-x = y(\ln|x| + c)$ Implicit sol

$y = \frac{-x}{\ln|x| + c}$. Explicit sol.

④ Bernoulli's equation

Standard form : $\boxed{y' + P(x)y = f(x)\, y^n}$ ①

Method of solving :

$\boxed{u = y^{1-n}}$

① becomes:

$\boxed{\underbrace{u' + (1-n)P(x)\,u = (1-n)f(x)}}$

First order linear eq

Integrating factor .. ...

$\boxed{u' = (1-n)\, y^{-n}\, y'}$

Multiply ① by $(1-n)y^{-n}$

$u' + P(x)y\,(1-n)y^{-n} = f(x)y^n \, (1-n)\, y^{-n}$

$u' + (1-n)\,P(x)\,y^{1-n} = (1-n)f(x)$

Ex : $y' + y = y^4$

$P(x) = 1, \quad f(x) = 1, \quad n = 4$

$$u = y^{1-4} = y^{-3}$$

Transform to

$$u' + (-3)\,1\,u = (-3)\,1$$

$$u' - 3u = -3$$

$$\mu = e^{\int -3\,dx} = e^{-3x} \Leftarrow$$

$$\int \underline{(\mu u)'} = \mu(-3) = \int e^{-3x}(-3)$$

$$(\mu u)' = \mu' u + \mu u'$$

$$= \mu u' + P(x) u \mu$$

$$\mu = e^{\int P(x)dx}$$

$$\mu u = e^{-3x} + C$$

$$y^{-3} = u = \frac{e^{-3x} + C}{e^{-3x}} = 1 + Ce^{3x}$$

$$y^{-3} = 1 + Ce^{3x}$$ Implicit solution

(f) $y' = f(Ax + By + C)$ (2)

$u = Ax + By + C$,

(2) becomes

$$u' = B f(u) + A \qquad (2^*)$$

Separable

$$u' = A + By'$$

Multiply (2) with B and add A

$$u' = B f(u) + A$$

Ex: $y' = \underbrace{e^{-(x+y)} - 1}_{f(x+y)}$ with $\boxed{y(0) = 0}$

$u = x + y$, $A = 1$, $B = 1$, $f(u) = e^{-u} - 1$

$$u' = 1(e^{-u} - 1) + 1$$

$$\frac{du}{dx} = u' = e^{-u} \Leftarrow \text{separable}$$

$$\int du\, e^{u} = \int dx$$

$$e^{u} = x + c$$

$$e^{x+y} = x + c \Leftarrow \text{general implicit sol}$$

$y(0) = 0$, $x = 0, y = 0$

$$e^{0} = 0 + c \Rightarrow c = e^{0} = 1.$$

$$\boxed{e^{x+y} = x + 1} \quad \text{Final answer}$$

Brief review of Equations we know how to solve

- Separable : separate variables
- First order linear equation : integrating factor
- Exact equation : Find $F(x,y)$ and set $F = C$
  Transform non-exact to exact : Another integrating factor
- Homogeneous functions : $(u = \frac{y}{x}) \Rightarrow$ Separable
- Bernoulli's equation : $(u = y^{1-n}) \Rightarrow$ First order linear
- $y' = f(Ax + By + C)$, $(u = Ax + By + C)$
  $\Rightarrow$ Separable equation.